

Horowitz-Randol pairs of curves in q -differential metrics

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Abstract: We examine equivalence relations on homotopy classes of curves coming from various spaces of flat metrics on a genus $g \geq 2$ surface. We prove an analog of a result of Randol (building on work of Horowitz) for subfamilies of flat metrics coming from q -differentials. In addition we also describe how these equivalence relations are related to each other.

1 Introduction

The work of Horowitz [10] and Randol [18] provides examples of the following: for every $n > 0$ there exist n distinct homotopy classes of curves $\gamma_1, \dots, \gamma_n$ on a closed oriented surface S such that for every hyperbolic metric m , $l_m(\gamma_i) = l_m(\gamma_j)$, for all i, j . In [12] Leininger studies this and related phenomenon for curves on S . He also asks whether other families of metrics exhibit similar behaviour.

Question 1.1. [12] *Do there exist pairs of distinct homotopy classes of curves γ and γ' which have the same length with respect to every metric in a given family of path metrics.*

For an arbitrary family of metrics one expects the answer to be no. Here we study this question for the metrics coming from q -differentials on S , for all $q \geq 1$. More precisely, let $Flat(S)$ denote the set of non-positively curved Euclidian cone metrics on S and $Flat(S, q)$ those that come from q -differentials. (See Section 2 for more details.) Let $\mathcal{C}(S)$ denote the set of homotopy classes of nontrivial closed curves on S . For every $q \in \mathbb{Z}_+$, define an equivalence relation on $\mathcal{C}(S)$ by declaring $\gamma \equiv_q \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$, $\forall m \in Flat(S, q)$.

In [12] Leininger answers Question 1.1 in the affirmative for metrics in $Flat(S, 2)$. In fact, writing $\gamma \equiv_h \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$, $\forall m$ hyperbolic, he proves:

Theorem 1.2. [12] *For every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_h \gamma' \Rightarrow \gamma \equiv_2 \gamma'$*

Consequently there are arbitrary large \equiv_2 - equivalence classes. In this paper we resolve Question 1.1 for all families $Flat(S, q)$, $q \geq 1$, proving \equiv_q is nontrivial. In fact there are arbitrary large \equiv_q - classes of curves.

Theorem 5.1. *For every $q_0, k \in \mathbb{Z}_+$ there are k distinct homotopy classes of curves $\gamma_1, \dots, \gamma_k \in \mathcal{C}(S)$ such that $\gamma_i \equiv_q \gamma_j, \forall i, j$ and $\forall q \leq q_0$. Thus $\forall q \in \mathbb{Z}_+$, the relation \equiv_q is non-trivial.*

However, in the limit this phenomenon disappears. To describe this, define $\gamma \equiv_\infty \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma'), \forall m \in Flat(S, q), \forall q \in \mathbb{Z}_+$.

Theorem 4.1. *The equivalence relation \equiv_∞ is trivial.*

Therefore, in general if $q \neq q'$, then \equiv_q and $\equiv_{q'}$ are different equivalence relations. However we have:

Theorem 3.1. *For every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_1 \gamma' \Leftrightarrow \gamma \equiv_2 \gamma'$*

In [12] it is also shown that the implication in Theorem 1.2 can not be reversed. As a consequence of our construction we see that a similar statement is true for any $q \in \mathbb{Z}_+$.

Theorem 5.5. *For every $q \in \mathbb{Z}_+$, $\exists \gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_q \gamma'$ but $\gamma \not\equiv_h \gamma'$.*

The outline of the paper is as follows. In Section 2 we define Euclidian cone and flat metrics, give standard definitions and state known theorems. Section 3 contains the proof of Theorem 3.1. In Section 4 we show that for every metric $m \in Flat(S)$ there is a sequence of metrics in $\bigcup_{q \in \mathbb{Z}_+} Flat(S, q)$ that converge to m and using this we prove that the equivalence relation \equiv_∞ is trivial [Theorem 4.1]. Finally, in Section 5 we describe constructions of curves reflecting properties of the metrics in $Flat(S, q)$ and we prove Theorem 5.1.

Notes: The construction of Horowitz and Randol is studied in [2], where a number of variants are also surveyed. These include similar results for hyperbolic 3-manifolds [13] and for metric graphs [11].

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2 Background

Let S denote a closed oriented surface of genus at least 2.

2.1 Euclidian Cone Metrics

A metric m on S is called a *Euclidian cone metric* if it satisfies the following properties:

- (i) m is a geodesic metric (not necessarily uniquely geodesic): the distance between 2 points is the length of a geodesic path between them.

- (ii) There is a finite set $X \subset S$ such that m on $S \setminus X$ is Euclidian, that is, locally isometric to \mathbb{R}^2 with the Euclidian metric.
- (iii) $(\forall x \in X)(\exists \epsilon > 0) B_\epsilon(x)$ is isometric to some cone. More precisely $B_\epsilon(x)$ is isometric to the metric space obtained by gluing together some (finite) number of sectors of ϵ -balls about 0 in \mathbb{R}^2 . Each x therefore has a well defined cone angle $c(x) \in \mathbb{R}_+$ which is the sum of the angles of the sectors used in construction. See Figure 1. For any $x \in S \setminus X$ we define $c(x) = 2\pi$.

See Figure 2 for an example of a Euclidian cone metric on a genus 2 surface.

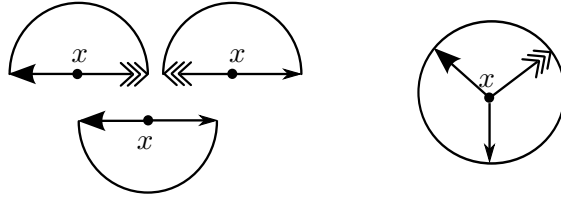


Figure 1: An example of a cone angle $c(x) = 3\pi$.

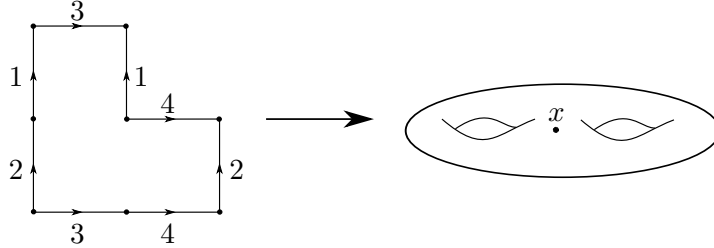


Figure 2: An example of a surface with Euclidian cone metric obtained by gluing sides of a polygon in \mathbb{R}^2 by translations as indicated. The result is a surface of genus 2 with a single cone point x with $c(x) = 6\pi$.

The *holonomy* homomorphism associated to m at any point $x \in S \setminus X$ is a homomorphism

$$\rho_x : \pi_1(S \setminus X, x) \rightarrow O(T_x(S \setminus X))$$

where $O(T_x(S \setminus X))$ is a group of orthogonal transformations of the tangent space of $S \setminus X$ at x . This is obtained by parallel translating a vector in $T_x(S \setminus X)$ along a loop in $S \setminus X$ based at x . Since our surface is oriented, the image is a subgroup of $SO(T_x(S \setminus X))$ - the group of rotations.

An orientation preserving isometry $\phi : T_x(S \setminus X) \rightarrow \mathbb{R}^2$ determines an isomorphism $SO(T_x(S \setminus X)) \rightarrow SO(2)$ independent of the choice of isometry ϕ . We therefore view the

holonomy homomorphism as a homomorphism to $SO(2)$. For a Euclidian cone metric m define $Hol = Hol(m) < SO(2)$ to be the image of the holonomy homomorphism.

We will construct Euclidian cone surfaces by gluing sides of polygons by maps $\{\rho_i \circ \tau_i\}_{i=1}^k$, which are compositions of translations τ_i and rotations ρ_i . Given $\gamma \in \pi_1(S \setminus X, x)$, $\rho_x(\gamma)$ is given by the composition of the rotations for the side gluings of the sides of the polygons crossed by γ . Therefore $Hol \leq \langle \rho_1, \rho_2, \dots, \rho_k \rangle$. See Figure 3.

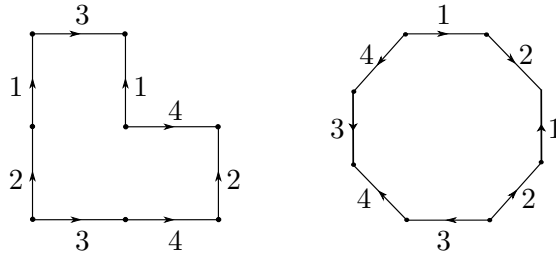


Figure 3: Genus 2 surfaces with $Hol = Id$ (on the left) and $Hol = \langle \rho_{\frac{\pi}{2}} \rangle$ (on the right).

In fact, we can obtain any Euclidian cone surface by gluing sides of a single *generalized Euclidian polygon* immersed in \mathbb{R}^2 . To explain, consider a triangulation of S by Euclidean triangles for which the vertex set is precisely the set of cone points. Such a triangulation exists by Theorem 4.4. in [14], for example (this is actually a Δ -complex structure as in [8] instead of a proper triangulation, but the distinction is unimportant). We get a dual graph of the triangulation constructed by defining a vertex for each triangle and an edge for each pair of triangles that share an edge. See Figure 4. This graph has a maximal tree. Now by cutting along the edges of the triangles whose dual edges do not belong in the maximal tree we get a simply connected surface that is a union of Euclidian triangles which isometrically immerses in the plane. This is the generalized Euclidian polygon, which we denote P . The surface S can be reconstructed from P by gluing pairs of edges. See Figure 5. If we glue P by translations and rotations $\{\rho_i \circ \tau_i\}_{i=1}^k$, then $Hol = \langle \rho_1, \rho_2, \dots, \rho_k \rangle$.

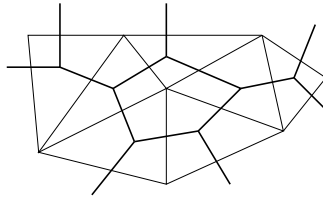


Figure 4: The dual graph of a triangulation.

Another important tool for us is the following well known fact:

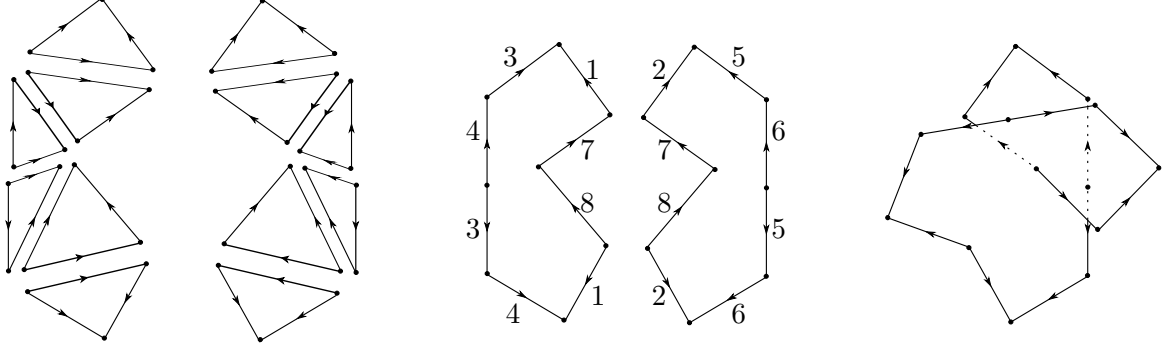


Figure 5: A genus 2 surface obtained by gluing triangles (on the left), two polygons (in the middle) and by gluing an immersed polygon (on the right).

Proposition 2.1. (Gauss-Bonnet formula) *Let R be a closed surface of genus $g \geq 0$ equipped with a Euclidian cone metric m . Then*

$$2\pi\chi(R) = \sum_{x \in X} (2\pi - c(x))$$

where X is the set of cone points.

We will give a proof of this proposition for completeness. A similar proof for compact surfaces with geometric structure of constant curvature can be found in [16].

Proof: Pick a triangulation of the surface R with triangles $\{\Delta_i\}_{i=1}^F$, so that cone points are the vertices of some of the triangles $\{\Delta_i\}$. Let the number of vertices of this triangulation be V , number of edges E and number of faces F . By Euler's theorem

$$V - E + F = \chi(R).$$

Each triangle has 3 edges, 2 edges are identified in pairs, therefore we get

$$3F = 2E.$$

Now we have

$$2V - 2E + 2F = 2\chi(R)$$

and therefore

$$2V - F = 2\chi(R).$$

This gives

$$2\pi V - \pi F = 2\pi\chi(R). \tag{1}$$

Denote $\alpha_i, \beta_i, \gamma_i$ the angles of Δ_i . Since all the triangles are Euclidian we have

$$\pi F = \sum_{i=1}^F (\alpha_i + \beta_i + \gamma_i).$$

On the other hand after gluing the triangles the sum of all angles in the triangles becomes the sum of angles around each vertex. Let \mathcal{U} denote the set of all vertices and $X \subset \mathcal{U}$ the set of all cone points. We then get:

$$\sum_{i=1}^F (\alpha_i + \beta_i + \gamma_i) = \sum_{x \in X} c(x) + \sum_{x \in \mathcal{U} \setminus X} 2\pi.$$

This gives us

$$\pi F = \sum_{x \in X} (c(x) - 2\pi) + \sum_{x \in \mathcal{U}} 2\pi = \sum_{x \in X} (c(x) - 2\pi) + 2\pi V. \quad (2)$$

From (1) and (2) we get our formula

$$2\pi\chi(R) = \sum_{x \in X} (2\pi - c(x)).$$

□

2.2 Flat Metrics

A Euclidian cone metric is called NPC (non-positively curved) if it is locally CAT(0). By the Gromov Link Condition this is equivalent to $c(x) \geq 2\pi$ for all $x \in S$ (See [5]). For example the surface in Figure 2 has one cone point and its angle is 6π , and so is NPC.

Define

$$Flat(S) = \{m \mid m \text{ is NPC Euclidian cone metric on } S\}.$$

By a metric on S we mean a metric inducing the given topology.

We are interested in the following class of metrics: For any $q \in \mathbb{Z}_+$ define

$$Flat(S, q) = \{m \in Flat(S) \mid Hol(m) < \langle \rho_{\frac{2\pi}{q}} \rangle\}$$

where ρ_θ is a rotation by angle θ .

Alternatively, $Flat(S, q)$ is the space of metrics coming from q -differentials on S . To explain this, suppose we are given a complex structure and a holomorphic q -differential

(see Chapter II of [6], for example). We can pick a small disk neighborhood U of any point $p_0 \in S$ with $\varphi(p_0) \neq 0$, containing no zeros of φ , and define preferred coordinates ζ for φ by

$$\zeta(p) = \int_{p_0}^p \sqrt[q]{\varphi}.$$

In these coordinates $\varphi = d\zeta^q$. Preferred coordinates give an atlas of charts on $S \setminus \{\text{zeros}(\varphi)\}$ to \mathbb{C} with transition functions of the form $T(z) = e^{\frac{2\pi i k}{q}} z + w$ for some $k \in \mathbb{Z}_+$, $w \in \mathbb{C}$. Pulling back the Euclidian metric we get Euclidian metric on $S \setminus \{\text{zeros}(\varphi)\}$. The completion of this metric is obtained by adding back in $\{\text{zeros}(\varphi)\}$ and at a zero of order k we have a cone angle $2\pi + \frac{2\pi k}{q}$. Therefore, the metric lies in $Flat(S, q)$. Conversely, take any metric in $Flat(S, q)$, choose local coordinates away from the singularities which are local isometries and so that the transition functions are translations and rotations by integer multiples of $\frac{2\pi}{q}$. Since these are holomorphic transformations preserving dz^q , this determines a complex structure and dz^q determines a holomorphic q -differential, and this extends over the singularities (compare with [15] and [19], for example, for the case $q = 2$).

2.3 Closed curves

Given a metric m and a homotopy class of curve $\gamma \in \mathcal{C}(S)$ we will define the length function: $l_m(\gamma) = \inf\{l_m(c) | c \in \gamma\}$. For every curve $\gamma \in \mathcal{C}(S)$ there is a geodesic representative on S due to Arzela–Ascoli theorem. Therefore $l_m(\gamma)$ is the length of its m -geodesic representative.

For m in $Flat(S)$, m -geodesics are closed Euclidian geodesics or a concatenation of Euclidian segments between cone points such that angles between consecutive segments are $\geq \pi$ on each side of the curve.

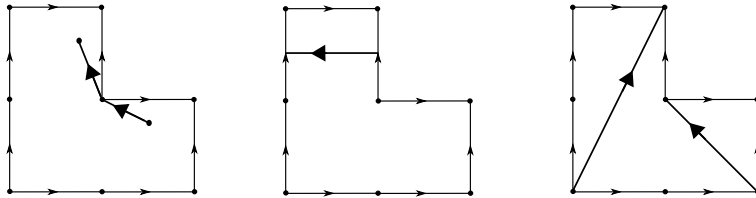


Figure 6: A geodesic through a cone point (on the left), a closed geodesic with no cone points (in the middle) and a closed self-intersecting geodesic containing a cone point (on the right). See Figure 2 for the gluings.

For two curves $\alpha, \beta \in \mathcal{C}(S)$ we can define the geometric intersection number, $i(\alpha, \beta) = \text{minimum number of double points of intersection of any two representatives of } \alpha \text{ and } \beta$. For hyperbolic metrics geodesic representatives realize geometric

intersection number.

Define $\mathcal{S}(S) \subset \mathcal{C}(S)$ to be the set of homotopy classes of simple closed curves on S .

A consequence of Thurston's [7], [17] and Bonahon's [4] work on measured laminations, closed curves and intersection number is the following:

Theorem 2.2. *Given any separating simple closed curve $\beta \in \mathcal{S}(S)$, there is a sequence $\{(t_n, \beta_n)\}_{n=1}^\infty \subset \mathbb{R}_+ \times \mathcal{S}(S)$ where each β_n is non-separating, so that for all $\alpha \in \mathcal{C}(S)$, $t_n i(\beta_n, \alpha) \rightarrow i(\beta, \alpha)$ as $n \rightarrow \infty$.*

We say that γ and $\gamma' \in \mathcal{C}(S)$ are simple intersection equivalent, $\gamma \equiv_{si} \gamma'$, if $i(\gamma, \alpha) = i(\gamma', \alpha)$ for every $\alpha \in \mathcal{S}(S)$. We will also need the next fact.

Theorem 2.3. [12] *For every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_h \gamma' \Rightarrow \gamma \equiv_{si} \gamma' \Leftrightarrow \gamma \equiv_2 \gamma'$.*

3 Relations \equiv_1 and \equiv_2

We now turn to the proof of

Theorem 3.1. *For every $\gamma, \gamma' \in \mathcal{C}(S)$, $\gamma \equiv_1 \gamma' \Leftrightarrow \gamma \equiv_2 \gamma'$*

Proof: Given $\gamma, \gamma' \in \mathcal{C}(S)$, we have

$$\gamma \equiv_{si} \gamma' \Rightarrow \gamma \equiv_2 \gamma' \Rightarrow \gamma \equiv_1 \gamma'$$

by Theorem 2.3 and the fact $Flat(S, 1) \subseteq Flat(S, 2)$. We want to prove

$$\gamma \equiv_1 \gamma' \Rightarrow \gamma \equiv_{si} \gamma'.$$

We claim that if $\gamma \equiv_1 \gamma'$, then $i(\alpha, \gamma) = i(\alpha, \gamma')$ for every non-separating curve α . Then if β is a separating curve, by Theorem 2.2 there is a sequence of non-separating curves β_n and positive real numbers t_n such that $t_n i(\gamma, \beta_n) \rightarrow i(\gamma, \beta)$ and $t_n i(\gamma', \beta_n) \rightarrow i(\gamma', \beta)$. Since $t_n i(\beta_n, \gamma) = t_n i(\beta_n, \gamma')$ by the claim, we will have $i(\beta, \gamma) = i(\beta, \gamma')$ for every separating simple closed curve β , hence every $\beta \in \mathcal{S}(S)$, and thus $\gamma \equiv_{si} \gamma'$.

For any $g \geq 2$, we can construct a closed genus g surface by gluing the arcs in the boundary of a cylinder so that in the resulting surface the core curve is non-separating. Moreover, this construction can be carried out on a Euclidian cylinder so that the resulting Euclidian cone metric has trivial holonomy. See Figure 7.

Suppose S has genus g and let α be a non-separating curve on S . Let X_g^ϵ be the surface of genus g obtained by gluing a rectangle as in Figure 7 with horizontal side

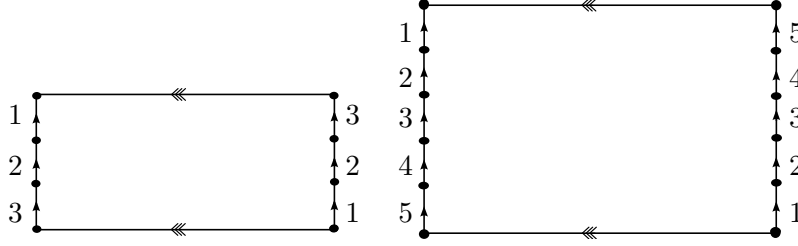


Figure 7: Examples of genus 2 and 3 surfaces. Gluing top sides to the bottom sides results in a cylinder, the rest of the gluing produces the closed surfaces.

length 1 and vertical side length $\epsilon > 0$. Let α_g be the core nonseparating curve of the cylinder See Figure 8. We assume that the obvious affine map from the 1×1 square to the $1 \times \epsilon$ rectangle descends to a homeomorphism $f_\epsilon : X_g^1 \rightarrow X_g^\epsilon$. Choose homeomorphisms $f : S \rightarrow X_g^1$ so that α is sent to α_g and let $h_\epsilon = f_\epsilon \circ f : S \rightarrow X_g^\epsilon$. Write $m_\alpha^\epsilon \in Flat(S, 1)$ to denote the metric obtained by pulling back via h_ϵ .

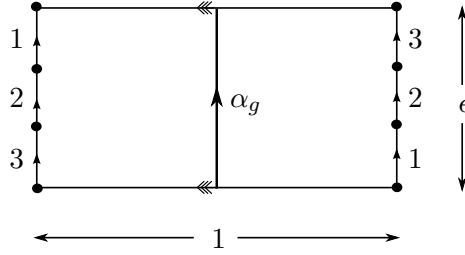


Figure 8: The surface X_2^ϵ and curve α_g .

Let γ be a curve on S . The geodesic representative of γ in m_α^ϵ is sent by h_ϵ to a union of straight lines running from one side of the rectangle to another and along the vertical boundary curves as in Figure 9.

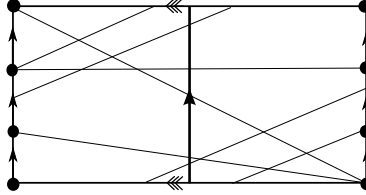


Figure 9: The geodesic representative of γ in m_α^ϵ . See Figure 8 for the gluings.

Thus we have

$$l_{m_\alpha^\epsilon}(\gamma) \geq i(\alpha, \gamma) \cdot 1 \quad (1)$$

since each geodesic segment that crosses the cylinder contributes 1 to intersection number and at least 1 to the length.

The curve γ is homotopic to a curve $\bar{\gamma}$ which is sent by h_ϵ to a union of straight line segments parallel to the side of length 1 of the rectangle and some segments of the vertical ϵ -length sides as in Figure 10.

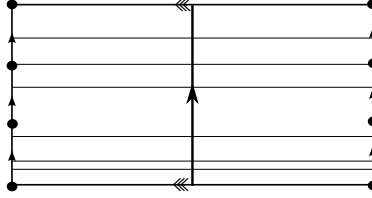


Figure 10: The representative $\bar{\gamma}$ in m_α^ϵ . See Figure 8 for the gluings.

From this we get

$$l_{m_\alpha^\epsilon}(\gamma) \leq \text{length}(\bar{\gamma}) \leq i(\alpha, \gamma) \cdot 1 + n_\gamma \epsilon \quad (2)$$

where n_γ is the number of vertical segments of $\bar{\gamma}$.

Now suppose $\gamma \equiv_1 \gamma'$. Then for every $\epsilon > 0$ we have $l_{m_\alpha^\epsilon}(\gamma) = l_{m_\alpha^\epsilon}(\gamma')$. By (1) and (2)

$$l_{m_\alpha^\epsilon}(\gamma) - n_\gamma \epsilon \leq i(\alpha, \gamma) \leq l_{m_\alpha^\epsilon}(\gamma)$$

and

$$l_{m_\alpha^\epsilon}(\gamma') - n_{\gamma'} \epsilon \leq i(\alpha, \gamma') \leq l_{m_\alpha^\epsilon}(\gamma').$$

Letting $\epsilon \rightarrow 0$ we get $i(\alpha, \gamma) = i(\alpha, \gamma')$, proving the claim.

Hence $\gamma \equiv_{si} \gamma'$, completing the proof. □

Corollary 3.2. $\forall q \in \mathbb{Z}_+, \quad \gamma \equiv_q \gamma' \Rightarrow \gamma \equiv_2 \gamma'.$

Proof: Since $\text{Flat}(S, 1) \subset \text{Flat}(S, q)$ for every q , we have

$$\gamma \equiv_q \gamma' \Rightarrow \gamma \equiv_1 \gamma' \Rightarrow \gamma \equiv_2 \gamma'.$$

□

4 The equivalence relation \equiv_∞

Next we will prove

Theorem 4.1. *The equivalence relation \equiv_∞ is trivial.*

To prove this we first prove a weaker statement. Define another equivalence relation on $\mathcal{C}(S)$ by declaring $\gamma \equiv_{\mathbb{R}} \gamma'$ if and only if $l_m(\gamma) = l_m(\gamma')$ for every $m \in \text{Flat}(S)$.

Theorem 4.2. *The equivalence relation $\equiv_{\mathbb{R}}$ is trivial.*

Proof: To prove this theorem we will show that for every $\gamma, \gamma' \in \mathcal{C}(S)$ there is $m \in \text{Flat}(S)$ so that $l_m(\gamma) \neq l_m(\gamma')$.

Let γ and $\gamma' \in \mathcal{C}(S)$. Pick a hyperbolic Riemannian metric g on S . If the geodesic representatives of γ and γ' have the same length in g , let $\epsilon > 0$ be small enough so that γ' does not intersect ϵ -ball around a point $x \in \gamma$. Let $\varphi : S \rightarrow \mathbb{R}$ be a smooth function so that $\varphi \equiv 0$ on $S \setminus B_\epsilon(x)$, $\varphi(x) = 1$ and $\varphi(S) = [0, 1]$. For $\delta > 0$ consider the Riemannian metric $g_\delta = g(1 - \delta\varphi)$. The metric g_δ is negatively curved for δ sufficiently small and the g_δ -length of γ is smaller than the g -length, while the length of γ' is not changed. Thus, for any two curves γ and $\gamma' \in \mathcal{C}(S)$ there is a negatively curved Riemannian metric m' such that $l_{m'}(\gamma) \neq l_{m'}(\gamma')$. In fact there are negatively curved metrics where there are no closed curves with the same length (see [18], [1], [3] for more general result).

Pick a geodesic triangulation of the surface S so that the geodesic representatives of γ and γ' are unions of edges of triangles. Being a locally $CAT(k)$ space, $k < 0$, implies also being locally $CAT(0)$. Now build a Euclidian cone metric m on S by replacing the triangles with Euclidian triangles with the same length sides. By the $CAT(0)$ property the angles of the Euclidian triangles are larger than the angles in the triangles they replaced. The vertices of the triangles are the only possible cone points. There are finitely many of them, and their angles are therefore $\geq 2\pi$. Hence $m \in \text{Flat}(S)$.

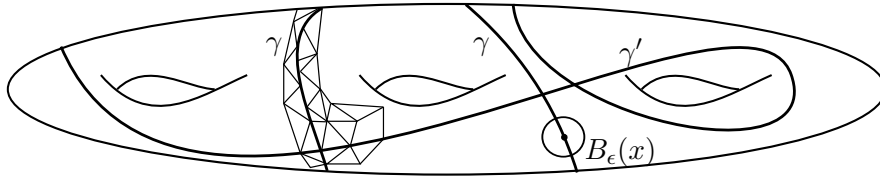


Figure 11: Part of a triangulation of a genus 3 surface with curves γ and γ' as unions of edges of the triangles.

The curves γ and γ' in the new metric become concatenations of Euclidian segments (sides of triangles). Since the sum of angles of a cone point on one side of the curve in

the metric m is greater than or equal to the sum of the angles in the metric m' we get that the angle around every cone point on each side of γ and γ' is $\geq \pi$. Therefore γ and γ' are also geodesic in the metric m .

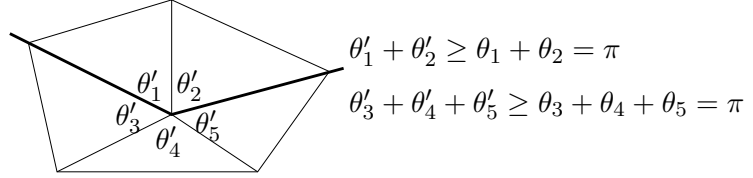


Figure 12: Euclidian triangles and a geodesic segment containing edges of the triangles.

We have

$$l_m(\gamma) = l_{m'}(\gamma) \neq l_{m'}(\gamma') = l_m(\gamma'),$$

and we proved our theorem. \square

Theorem 4.3. $\overline{\bigcup_{q \in \mathbb{Z}_+} Flat(S, q)} = Flat(S)$. More precisely, for every $m \in Flat(S)$, $\exists m_n \in \bigcup_{q \in \mathbb{Z}_+} Flat(S, q)$, so that $id : (S, m_n) \rightarrow (S, m)$ is K_n -bilipschitz and $K_n \rightarrow 1$ as $n \rightarrow \infty$.

Proof: Let $m \in Flat(S)$. Take a triangulation of S by Euclidean triangles with all vertices being cone points. Using this triangulation, view S as obtained from a generalized Euclidian polygon P isometrically immersed in \mathbb{R}^2 by gluing the edges in pairs by isometries as in Section 2. Orient the edges s_i using the boundary orientation coming from the immersion into \mathbb{R}^2 . Let z_i be a complex number that represents the edge s_i and θ_i the argument of z_i . We have $z_1 + z_2 + \dots + z_{2n} = 0$. Assume, by rotating if necessary, that $\theta_1 = 0$ and by relabeling if necessary that z_{2i-1} and z_{2i} correspond to the edges that are glued together, $\forall i = 1, \dots, n$. We have $|z_1| = |z_2|, |z_3| = |z_4|, \dots, |z_{2n-1}| = |z_{2n}|$. See Figure 13.

If $z_{2i-1} = -z_{2i}, \forall i = 1, \dots, n$ then $Hol(S) = \{Id\}$ and therefore $m \in Flat(S, 1)$ and we set $m_n = m$ for all n .

Therefore we assume that there is some i so that $z_{2i-1} \neq -z_{2i}$. After changing indices if necessary we may assume $z_{2n-1} \neq -z_{2n}$. Fix $0 < \epsilon < \frac{|z_{2n-1} + z_{2n}|}{2}$. We will construct an isometrically immersed polygon P_ϵ so that the angles the sides of P_ϵ make with the real axis are all in $\mathbb{Q}\pi$, and $P_\epsilon \rightarrow P$ as $\epsilon \rightarrow 0$.

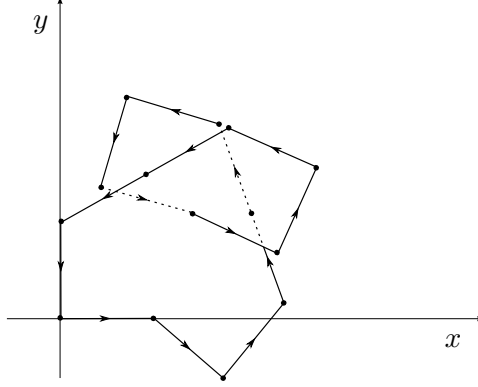


Figure 13: An immersed polygon in \mathbb{R}^2 with one edge on the x -axis.

Denote $\tau = -z_1 - \dots - z_{2n-2}$. Since $\tau = z_{2n-1} + z_{2n}$ it follows that $|\tau| > 2\epsilon$. Let $\tilde{z}_1 = z_1$. For $2 \leq j \leq 2n-4$ choose \tilde{z}_j so that $|z_j| = |\tilde{z}_j|$, $\tilde{\theta}_j \in \mathbb{Q}\pi$ and $|z_j - \tilde{z}_j| < \frac{\epsilon}{2n}$. Now, choose $\tilde{z}_{2n-3}, \tilde{z}_{2n-2}$ so that $|\tilde{z}_{2n-3}| = |\tilde{z}_{2n-2}|$, $\tilde{\theta}_{2n-3}, \tilde{\theta}_{2n-2} \in \mathbb{Q}\pi$, $|z_i - \tilde{z}_i| < \frac{\epsilon}{2n}$ for $i = 2n-3, 2n-2$, and also so that the angle of $\tilde{z}_1 + \dots + \tilde{z}_{2n-3} + \tilde{z}_{2n-2}$ is in $\mathbb{Q}\pi$. We arrange this in the following way: First construct z'_i , $i = 2n-3, 2n-2$, so that their arguments are in $\mathbb{Q}\pi$ and $|z'_i| = |z_i|$, $|z'_i - z_i| < \frac{\epsilon}{4n}$ for $i = 2n-3, 2n-2$. Then construct \tilde{z}_i , $i = 2n-3, 2n-2$, so that $\text{Arg}(z'_i) = \text{Arg}(\tilde{z}_i)$, $|z'_i - \tilde{z}_i| < \frac{\epsilon}{4n}$ and the angle of $\tilde{z}_1 + \dots + \tilde{z}_{2n-3} + \tilde{z}_{2n-2}$ is in $\mathbb{Q}\pi$. See Figure 14.

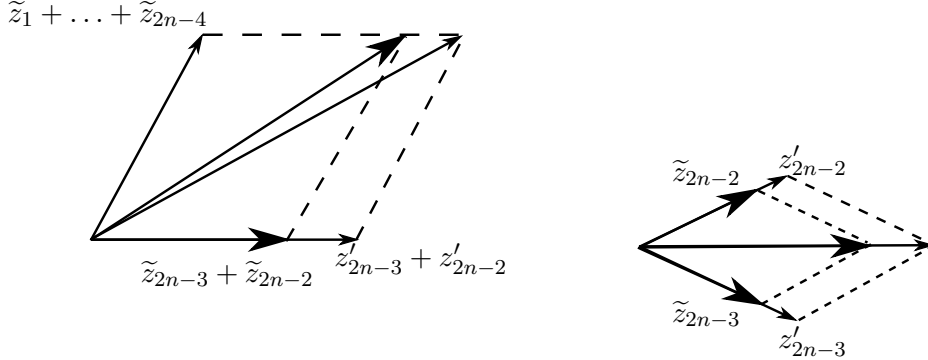


Figure 14: Changing the angle of $\tilde{z}_1 + \dots + \tilde{z}_{2n-3} + \tilde{z}_{2n-2}$ keeping $|\tilde{z}_{2n-3}| = |\tilde{z}_{2n-2}|$.

Denote $\tilde{\tau} = -\tilde{z}_1 - \dots - \tilde{z}_{2n-2}$. By the triangle inequality $|\tau - \tilde{\tau}| \leq \sum_{j=1}^{2n-2} |z_j - \tilde{z}_j| < \epsilon$. Observe that $\tilde{\tau} \neq 0$, because if $\tilde{\tau} = 0$ then $2\epsilon < |\tau| < \epsilon$, which is a contradiction.

Since $\tau \neq 0$, the point z_{2n-1} is on the perpendicular bisector l of the vector τ . Let

\tilde{l} be the perpendicular bisector of the vector $\tilde{\tau}$. Let $\delta = 2\text{dist}(z_{2n-1}, \tilde{l})$. Note that as $\epsilon \rightarrow 0$, $\delta \rightarrow 0$. Now choose $\tilde{z}_{2n-1} \in \tilde{l} \cap B(z_{2n-1}, \delta)$ so that the argument of \tilde{z}_{2n-1} is in $\mathbb{Q}\pi$. See Figure 15. Set $\tilde{z}_{2n} = \tilde{\tau} - \tilde{z}_{2n-1}$. Since \tilde{z}_{2n-1} is on the perpendicular bisector of $\tilde{\tau}$ then $|\tilde{z}_{2n-1}| = |\tilde{z}_{2n}|$ and since $\text{Arg}(\tilde{\tau}) = \frac{1}{2}(\theta_{2n-1} + \theta_{2n})$, it follows that $\theta_{2n} \in \mathbb{Q}\pi$.

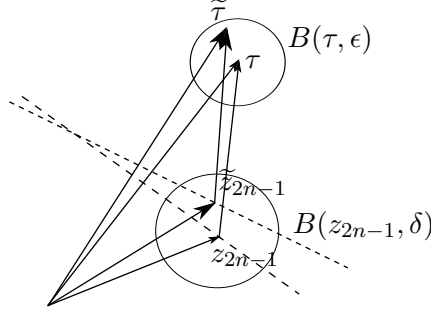


Figure 15: Constructing the vector \tilde{z}_{2n-1} .

We have $\tilde{z}_1 + \dots + \tilde{z}_{2n} = 0$ and $\tilde{\theta}_i \in \mathbb{Q}\pi$ for every i . Therefore for $\epsilon > 0$ sufficiently small $\{\tilde{z}_i\}_{i=1}^{2n}$ determines an isometrically immersed polygon P_ϵ with all sides having angles being rational multiples of π .

For sufficiently small ϵ there is a bilipschitz homeomorphism $f_\epsilon : P \rightarrow P_\epsilon$, which is linear on edges. For example, when $\epsilon > 0$ is sufficiently small, we can take a triangulation of P_ϵ with the same combinatorics as the one used to construct P . Map each triangle of P linearly to the corresponding triangle of P_ϵ . That way we get a map f_ϵ which is linear, and thus bilipschitz, on each triangle. It follows that f_ϵ is bilipschitz on P and as $\epsilon \rightarrow 0$ bilipschitz constant $K(f_\epsilon) \rightarrow 1$. The surface S is obtained by gluing the polygon P , so $S = P/\sim$ where $t \sim \varphi(t)$ and $\varphi : \cup s_i \rightarrow \cup s_i$ is the gluing map. Define $\varphi_\epsilon = f_\epsilon \varphi f_\epsilon^{-1} : \cup f_\epsilon(s_i) \rightarrow \cup f_\epsilon(s_i)$. The map f_ϵ descends to $\bar{f}_\epsilon : P/\sim \rightarrow P_\epsilon/\sim_\epsilon$ where $t \sim_\epsilon \varphi_\epsilon(t)$. Define a Euclidian cone metric m_ϵ on S by pulling back the metric on P_ϵ/\sim_ϵ by \bar{f}_ϵ . The holonomy of m_ϵ is generated by rotations through angles in $\mathbb{Q}\pi$. Therefore there is $q_\epsilon \in \mathbb{Z}_+$ so that $m_\epsilon \in \text{Flat}(S, q_\epsilon)$ and $m_\epsilon \rightarrow m$ as $\epsilon \rightarrow 0$. \square

Theorem 4.1 now follows easily from the previous theorem.

Proof of Theorem 4.1: Suppose that we are given curves $\gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_\infty \gamma'$. By Theorem 4.3 it follows that $\forall m \in \text{Flat}(S)$, there is a sequence of metrics $\{m_n \in \text{Flat}(S, q_n)\}_{n=1}^\infty$ such that $\text{id} : (S, m_n) \rightarrow (S, m)$ is K_n -bilipschitz and $K_n \rightarrow 1$ as $n \rightarrow \infty$. Thus $\lim_{n \rightarrow \infty} l_{m_n}(\gamma) = l_m(\gamma)$ and $\lim_{n \rightarrow \infty} l_{m_n}(\gamma') = l_m(\gamma')$. Since $l_{m_n}(\gamma) = l_{m_n}(\gamma')$ by assumption then $l_m(\gamma) = l_m(\gamma')$. That implies $\gamma \equiv_{\mathbb{R}} \gamma'$ which is a contradiction by Theorem 4.2. \square

In contrast to Theorem 1.2 we observe the following.

Corollary 4.4. *There exists $q \in \mathbb{Z}_+$ and curves $\gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_h \gamma'$ but $\gamma \not\equiv_q \gamma'$.*

Proof: Assume that $\forall q \in \mathbb{Z}_+$ and $\forall \gamma, \gamma' \in \mathcal{C}(S) \gamma \equiv_h \gamma' \Rightarrow \gamma \equiv_q \gamma'$. Then for all γ and $\gamma', \gamma \equiv_h \gamma' \Rightarrow \gamma \equiv_\infty \gamma'$ which is a contradiction. \square

Using the same idea as in Theorem 4.3 we can prove a stronger statement.

Theorem 4.5. *For every infinite sequence of distinct positive integers $\{q_i\}_{i=1}^\infty$, $\overline{\bigcup_{i=1}^\infty \text{Flat}(S, q_i)} = \text{Flat}(S)$.*

Proof: The proof of this theorem follows the proof of Theorem 4.3 if we approximate angles θ_j with angles $\tilde{\theta}_j \in \mathbb{Z} \frac{2\pi}{q_l}$ for appropriately chosen $q_l \gg 0$. \square

5 $\text{Flat}(S, q)$

In this section we prove:

Theorem 5.1. *For every $q_0, k \in \mathbb{Z}_+$ there are k distinct homotopy classes of curves $\gamma_1, \dots, \gamma_k \in \mathcal{C}(S)$ such that $\gamma_i \equiv_q \gamma_j, \forall i, j$ and $\forall q \leq q_0$. Thus $\forall q \in \mathbb{Z}_+$, the relation \equiv_q is non-trivial.*

The proof uses the following two lemmas.

Lemma 5.2. *For any $q_0 \in \mathbb{Z}_+$, there is a curve $\gamma \in \mathcal{C}(S)$ such that for every $m \in \text{Flat}(S, q), q \leq q_0$ the geodesic representative γ_m of γ contains a cone point and $[\gamma] \neq 0$ in $H_1(S)$.*

Proof: Fix a hyperbolic metric on S and identify the universal cover as $\mathbb{H}^2 \rightarrow S$. Fix $q_0 \in \mathbb{Z}_+$, and take $q_0 + 1$ geodesics $\alpha_1, \dots, \alpha_{q_0+1}$ in \mathbb{H}^2 meeting at 0. See Figure 16.

By ergodicity of the geodesic flow, there is a dense geodesic on S [9]. Given $\epsilon > 0$, we can therefore construct an ϵ -dense closed geodesic.

From this it follows that we can construct a closed geodesic γ on S with lifts $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{q_0+1}$ having tangent vectors within ϵ distance from the tangent vectors of $\alpha_1, \dots, \alpha_{q_0+1}$ at 0. For ϵ sufficiently small the end points of $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{q_0+1}$ pairwise link in the same pattern as $\alpha_1, \dots, \alpha_{q_0+1}$. In particular, every pair $\tilde{\alpha}_i, \tilde{\alpha}_j$ intersects. For every $m \in \text{Flat}(S, q)$ and m -geodesic representative γ_m of γ in m there are lifts $(\tilde{\gamma}_1)_m, \dots, (\tilde{\gamma}_{q_0+1})_m$ which are quasi-geodesics in \mathbb{H}^2 , with the same endpoints as $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{q_0+1}$, and hence with endpoints that all pairwise link. See Figure 17.

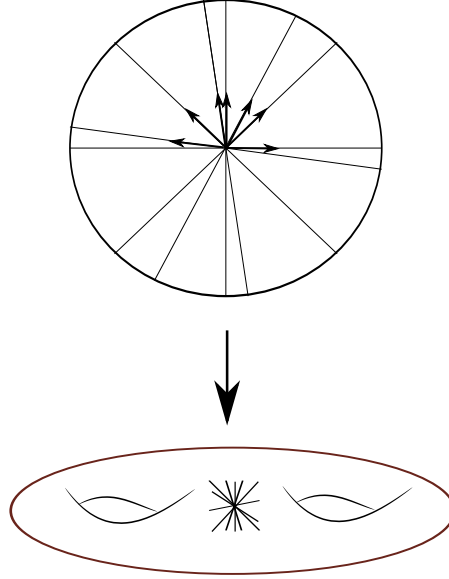


Figure 16: Projecting $\alpha_1, \dots, \alpha_{q_0+1}$ into the surface S .

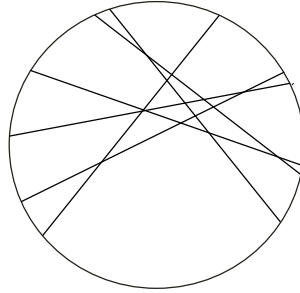


Figure 17: Lifts of γ_m to \mathbb{H}^2 .

If $[\gamma] = 0$ in $H_1(S)$, we replace γ with a different curve as follows. Let δ be any curve with $[\delta] \neq 0$ in $H_1(S)$ that intersects γ . Construct a new curve which runs n times around γ , then at the intersection point switches and runs once around δ . If n is large enough, this constructs a curve which maintains the property of having $q_0 + 1$ lifts with endpoints that pairwise link. This new curve is homologous to δ , and so we replace γ with this curve.

Now assume that there is $m \in \text{Flat}(S, q)$, $q \leq q_0$ so that γ_m does not go through a cone point. Then there is an isometrically immersed Euclidian cylinder $S^1 \times [0, a] \rightarrow S$, for some $a > 0$, such that the image of $S^1 \times \{\frac{a}{2}\}$ is the geodesic γ_m and the image of $S^1 \times [0, a]$ does not contain any cone points. It follows that γ_m has only finitely many

transverse self intersecting points. Because the holonomy group of m is in $\langle \rho_{\frac{2\pi}{q}} \rangle$, the angles of intersections are integer multiples of $\frac{2\pi}{q}$.

Lifts of γ_m have lifted cylinders which are strips isometric to $\mathbb{R} \times [0, a]$ in the universal cover. We consider the $q_0 + 1$ lifts $(\tilde{\gamma}_1)_m, \dots, (\tilde{\gamma}_{q_0+1})_m$ as above and fix one of the lifts $(\tilde{\gamma}_1)_m$ and a strip about it. Let $P \in (\tilde{\gamma}_1)_m \cap (\tilde{\gamma}_2)_m$ be the point of intersection of $(\tilde{\gamma}_2)_m$ with $(\tilde{\gamma}_1)_m$. For every $j \geq 3$ parallel transport the tangent vector to $\tilde{\gamma}_j$ at the point of intersection $(\tilde{\gamma}_1)_m \cap (\tilde{\gamma}_j)_m$ inside the strip to P . The point P is not a cone point and there are no cone points inside the strip. If $(\tilde{\gamma}_1)_m, (\tilde{\gamma}_i)_m, (\tilde{\gamma}_j)_m$ do not intersect in a common point then they form a triangle in \tilde{S} . Since \tilde{S} is a $CAT(0)$ space the angle sum of a triangle is less than π , and hence $(\tilde{\gamma}_i)_m$ and $(\tilde{\gamma}_j)_m$ cannot make the same angle with $(\tilde{\gamma}_1)_m$. Hence no two tangent vectors get transported to the same vector. See Figure 18.

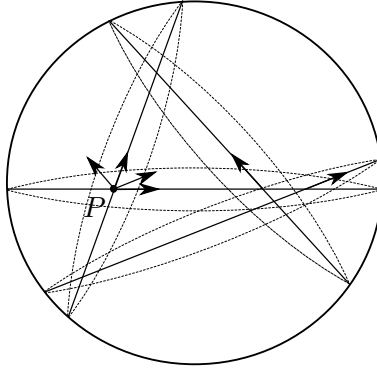


Figure 18: Strips in the universal cover.

By the holonomy condition all angles of intersections are integer multiples of $\frac{2\pi}{q}$, which is greater than or equal to $\frac{2\pi}{q_0}$. On the other hand we have $q_0 + 1$ vectors based at P , no two of them equal, therefore there is a pair of vectors with the angle between them less than $\frac{2\pi}{q_0}$. It follows that for some j , $(\tilde{\gamma}_1)_m$ and $(\tilde{\gamma}_j)_m$ make an angle which is not an integer multiple of $\frac{2\pi}{q_0}$. This is a contradiction and thus proves that γ_m has to contain at least one cone point for every $m \in Flat(S, q)$, $q \leq q_0$. \square

Lemma 5.3. *Let γ be the curve constructed in Lemma 5.2, and let m be a metric in $Flat(S, q)$. Given two lifts $\tilde{\gamma}, \tilde{\gamma}_0$ with linking endpoints and stabilizers generated by h and g , respectively, let $w \in \tilde{\gamma} \cap \tilde{\gamma}_0$. Then the m -geodesic from $h^{(q+2)}(w) \in \tilde{\gamma}$ to any point along $\tilde{\gamma}_0$ must run along a positive length segment of $\tilde{\gamma}$. Moreover, this geodesic meets $\tilde{\gamma}_0$ on $[g^{-(q+2)}(w), g^{(q+2)}(w)] \subset \tilde{\gamma}_0$.*

Proof: It could happen that $l_m(\tilde{\gamma}_0 \cap \tilde{\gamma}) > 0$ as in Figure 19. We first claim that $l_m(\tilde{\gamma}_0 \cap \tilde{\gamma}) < l_m(\gamma)$. To show this let v, u be cone points at each end of $\tilde{\gamma}_0 \cap \tilde{\gamma}$.

If $l_m(\tilde{\gamma}_0 \cap \tilde{\gamma}) \geq l_m(\gamma)$ then since the translation length of g and h is equal to $l_m(\gamma)$, we have $g(v) = h(v)$ or $g^{-1}(v) = h(v)$ which is a contradiction as $\pi_1(S)$ acts freely on \tilde{S} .

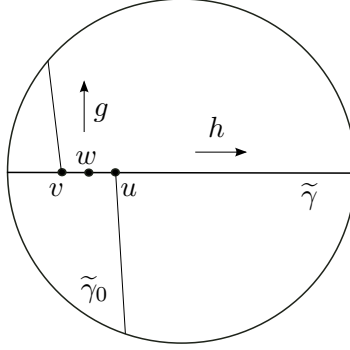


Figure 19: Two lifts of γ to the universal cover.

Now let v_1, \dots, v_n be all the different cone points along $[h^{(q+1)}w, h^{(q+2)}(w)) \subset \tilde{\gamma}$ and let θ_i be the angles $\tilde{\gamma}$ makes at v_i all on one side of $\tilde{\gamma}$, and let θ'_i be all the angles at v_i on the other side. Observe that there is some i so that $\theta_i > \pi$. Otherwise there is a small neighborhood of one side of $\tilde{\gamma}$ with no cone points and we can find a geodesic homotopic to γ with no cone points by doing a straight line homotopy as shown in Figure 20. Similarly, there is j so that $\theta'_j > \pi$.

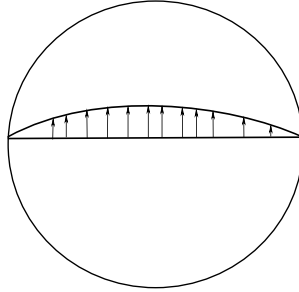


Figure 20: Homotopy.

Let $v = v_i$ with $\theta_i > \pi$ and $v' = v_j$ with $\theta'_j > \pi$. Let σ be the continuation of the geodesic segment $[h^{(q+2)}(w), v] \subset \tilde{\gamma}$ to a geodesic ray starting at $h^{(q+2)}(w)$ and at every cone point past v having angle π on the right. See Figure 21. We define σ' similarly as the continuation of $[h^{(q+2)}(w), v'] \subset \tilde{\gamma}$ with angle π on the left.

Claim: The rays σ and σ' do not intersect $\tilde{\gamma}_0$.

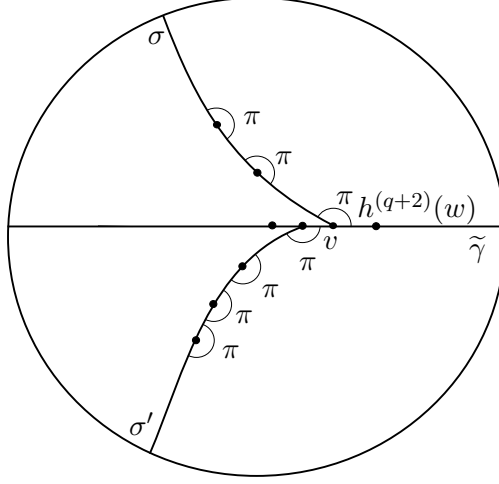


Figure 21: Geodesic rays as continuations of the geodesic segment γ .

Proof: Assume that $\sigma \cap \tilde{\gamma}_0 \neq \emptyset$. Then $\tilde{\gamma}$, $\tilde{\gamma}_0$ and σ form a triangle in \tilde{S} . Denote the angles of the triangle by β_1 , β_2 and β_3 . See Figure 22.

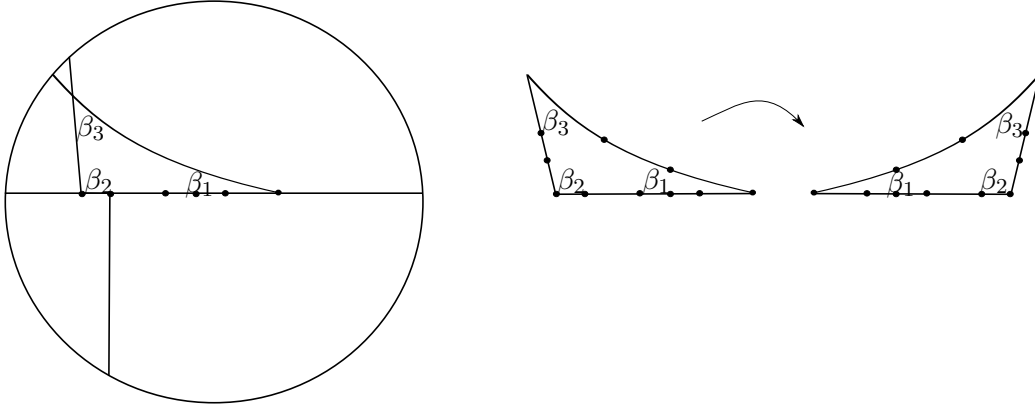


Figure 22: Gluing two triangles to get a sphere.

Take two copies of the given triangle and glue them together to get a sphere R with an induced Euclidian cone metric. See Figure 22. The Euler characteristic of R is 2,

and by the Gauss-Bonnet formula (see Proposition 2.1)

$$-2\pi\chi(R) = \sum_{x \in X} (c(x) - 2\pi),$$

where X is the set of all cone points of R . Three cone points come from the triangle vertices and they have cone angles $2\beta_l$, $l = 1, 2, 3$. The rest come from cone points along the sides or the inside of the triangles and have cone angles $> 2\pi$. In particular we have cone points $h^{-1}(v_j), \dots, h^{-q}(v_j)$ with cone angles $2\theta_j$, $j = 1, \dots, n$. Therefore we get the following inequality:

$$-4\pi = \sum_{x \in X} (c(x) - 2\pi) \geq \sum_{i=1}^q \sum_{j=1}^n (2\theta_j - 2\pi) + (2(\beta_1 + \beta_2 + \beta_3) - 6\pi).$$

It follows that

$$\pi - (\beta_1 + \beta_2 + \beta_3) \geq \sum_{i=1}^q \sum_{j=1}^n (\theta_j - \pi).$$

The holonomy for every metric in $Flat(S, q)$ is a rotation through an angle of the form $\frac{2\pi}{q}k$, $k \in \mathbb{Z}$. Therefore we get $\sum_{j=1}^n (\theta_j - \pi) = \frac{2\pi}{q}k$ for some integer $k > 0$, and so

$$\sum_{j=1}^n (\theta_j - \pi) \geq \frac{2\pi}{q}.$$

It follows that $\sum_{i=1}^q \sum_{j=1}^n (\theta_j - \pi) = q \sum_{j=1}^n (\theta_j - \pi) \geq q \frac{2\pi}{q} = 2\pi$, and thus:

$$\pi - (\beta_1 + \beta_2 + \beta_3) \geq 2\pi$$

which is a contradiction. Therefore $\sigma \cap \tilde{\gamma}_0 = \emptyset$. The same argument shows $\sigma' \cap \tilde{\gamma}_0 = \emptyset$. This proves the claim.

Now we'll show that every geodesic from $h^{(q+2)}(w)$ to $\tilde{\gamma}_0$ has to share a positive length geodesic segment with $\tilde{\gamma}$.

There is a unique geodesic between any 2 distinct points in \tilde{S} , since it is a $CAT(0)$ space. Let ζ be a geodesic from $h^{(q+2)}(w)$ to some point on $\tilde{\gamma}_0$. Let $\sigma_0 = \sigma - \tilde{\gamma}$ and $\sigma'_0 = \sigma' - \tilde{\gamma}$. Assume ζ does not share a positive length geodesic segment of $\tilde{\gamma}$. By the previous claim ζ must intersect one of σ_0 and σ'_0 . See Figure 23. Without loss of generality assume that $A \in \zeta \cap \sigma_0$ is such a point. The path from $h^{(q+2)}(w)$ to v following the geodesic segments on $\tilde{\gamma}$ and then from v to A along σ_0 is a geodesic by construction since all cone angles are $\geq \pi$ on both sides of the path. Since the initial arc of ζ to A provides a different geodesic, this contradicts the uniqueness of geodesics

in \tilde{S} and we can see that ζ cannot cross either of σ_0 or σ'_0 and hence must share a positive length segment with $\tilde{\gamma}$.

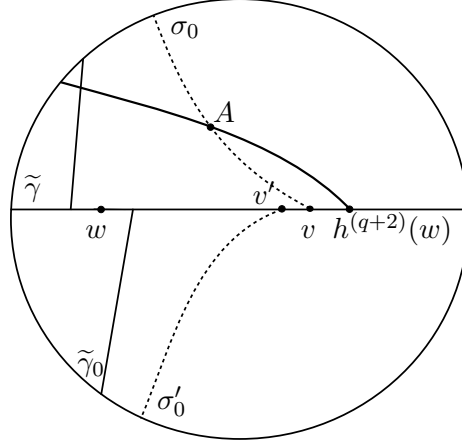


Figure 23: A geodesic segment from $\tilde{\gamma}$ to $\tilde{\gamma}_1$.

For the same reason the geodesics from $g^{(q+2)}(w)$ or $g^{-(q+2)}(w)$ to $\tilde{\gamma}$ must share a positive length segment with $\tilde{\gamma}_0$. Therefore every geodesic from $h^{(q+2)}(w)$ to a point on $\tilde{\gamma}_0$ meets $\tilde{\gamma}$ on $[g^{-(q+2)}(w), g^{(q+2)}(w)]$. \square

Now we can prove the main theorem.

Proof of Theorem 5.1: Let $\gamma \in \mathcal{C}(S)$ be as in Lemma 5.2. Let $\tilde{\gamma}$ and $\tilde{\gamma}_0$ be lifts of γ with endpoints that link. Suppose the stabilizer of $\tilde{\gamma}$ is generated by h . Let $\tilde{\gamma}_{-1}$ and $\tilde{\gamma}_1$ be defined by $\tilde{\gamma}_i = h^{i(q+2)}(\tilde{\gamma}_0)$, $i = \pm 1$. See Figure 24. Denote h_{-1} and h_1 conjugate elements of $\pi_1(S)$ that generate the stabilizers of $\tilde{\gamma}_{-1}$ and $\tilde{\gamma}_1$ respectively.

Let $F(a, b)$ denote the free group on two generators a and b . Define $\phi : F(a, b) \rightarrow \pi_1(S)$ by $a \rightarrow h_{-1}^{2(q+2)}$ and $b \rightarrow h_1^{2(q+2)}$.

Claim: The ϕ -images of the words

$$w_0 = (ab)^k, w_1 = (ab)^{k-1}(ab^{-1}), \dots, w_{k-1} = (ab)(ab^{-1})^{k-1}$$

represent distinct elements in $\mathcal{C}(S)$ and have the same length in every metric $m \in \text{Flat}(S, q)$, $q \in \mathbb{Z}_+$.

Let $m \in \text{Flat}(S, q)$. Let $(\tilde{\gamma}_i)_m, (\tilde{\gamma})_m$ be corresponding lifts of the m -geodesic representatives of γ . We have $(\tilde{\gamma}_1)_m = h^{2(q+2)}((\tilde{\gamma}_{-1})_m)$. Let z' be a point on $(\tilde{\gamma})_m$ so that

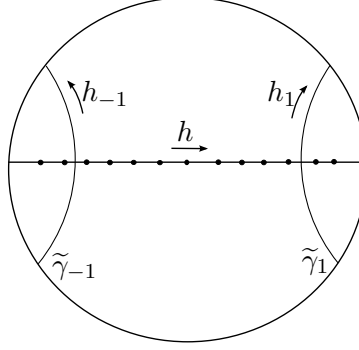


Figure 24: Lifts of γ .

$(\tilde{\gamma}_i)_m \cap (\tilde{\gamma})_m \in [h^{i(q+1)}(z'), h^{i(q+2)}(z')]$, $i = \pm 1$. From Lemma 5.3 we know that every geodesic from z' to $(\tilde{\gamma}_i)_m$, $i = \pm 1$, has to run along positive line segment of $(\tilde{\gamma})_m$ and meet $(\tilde{\gamma}_i)_m$ on $[h_i^{-(q+2)}(z'), h_i^{(q+2)}(z')]$.

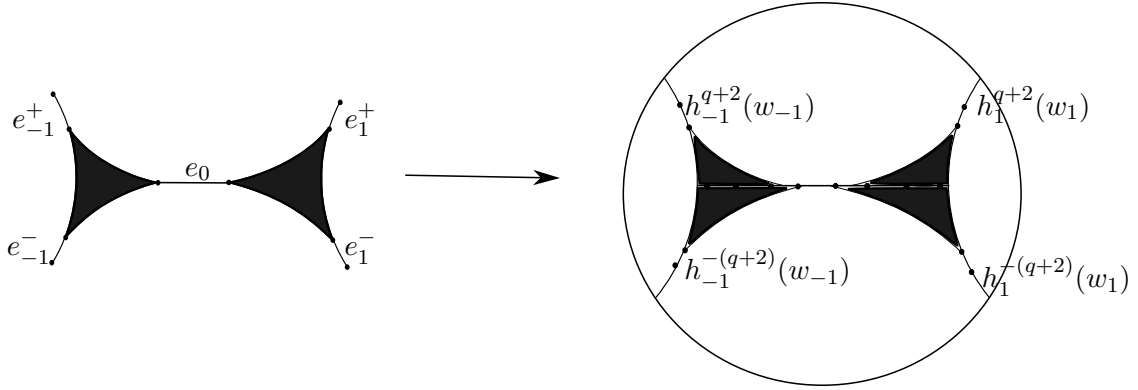


Figure 25: The m -convex hull of $(\tilde{\gamma}_{-1})_m$ and $(\tilde{\gamma}_1)_m$ on the right, and the 2-complex Γ' on the left.

It follows that the m -convex hull of $(\tilde{\gamma}_{-1})_m$ and $(\tilde{\gamma}_1)_m$ consists of $(\tilde{\gamma}_{-1})_m \cup (\tilde{\gamma}_1)_m$ together with an arc of $(\tilde{\gamma})_m$ and two (possibly degenerate) triangles. Choose a point $w_i \in (\tilde{\gamma})_m \cap (\tilde{\gamma}_i)_m$ for $i = \pm 1$ and consider the point $h_i^{\pm(q+2)}(w_i)$ along $(\tilde{\gamma}_i)_m$. Let Γ' be the metric 2-complex with two (possibly degenerate) triangles and five edges determined by $h_i^{\pm(q+2)}(w_i)$ as shown in the Figure 25 and view this as mapping into \tilde{S} . Let Γ denote the metric 2-complex obtained by gluing $h_i^{-(q+2)}(w_i)$ to $h_i^{(q+2)}(w_i)$, for $i = \pm 1$. The inclusion $\Gamma' \rightarrow \tilde{S}$ descends to a locally convex, local isometry $f : \Gamma \rightarrow S$.

Identifying $\pi_1(\Gamma) = F(a, b)$ as shown in Figure 26 we have $f_* = \phi$.

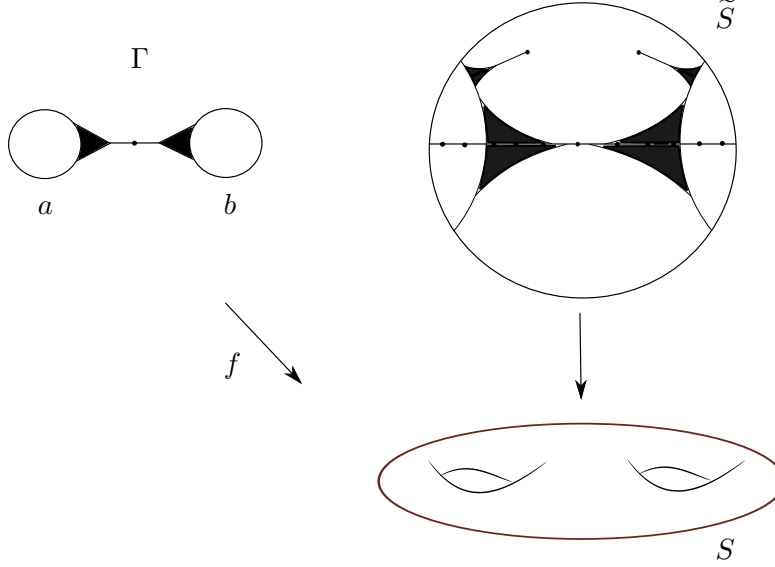


Figure 26: Constructing 2-complex Γ from the curve γ .

By the previous construction $[f_*(a)] = [f_*(b)] = \pm 2(q+2)[\gamma]$ in $H_1(S)$. It follows that $[f_*(w_j)] = \pm(2k-2j)[\gamma]$. By construction $[\gamma] \neq 0$, thus $\{f_*(w_j)\}_{j=0}^{k-1}$ are distinct classes of non-null homotopic curves. Since f is a locally convex, local isometry, by measuring lengths on Γ instead of S we can see $l_m(f_*(w_j)) = l_m(f_*(w_i))$ for every $i, j = 0, \dots, k-1$. \square

Corollary 5.4. *Let $q_1, q_2 \in \mathbb{Z}_+$. If $q_1|q_2$ then $\equiv_{q_2} \Rightarrow \equiv_{q_1}$. The reverse implication is not true in general.*

Proof: If $q_1|q_2$ then $\text{Flat}(S, q_1) \subset \text{Flat}(S, q_2)$. Thus the first part of the statement follows.

Assume the reverse statement is true for all $q_1|q_2$. Let $q \in \mathbb{Z}_+$. Then $\equiv_q \Rightarrow \equiv_{q^i}$ for every positive integer i . By Theorem 5.1 there are two distinct curves $\gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_q \gamma'$. By assumption we get $\gamma \equiv_{q^i} \gamma', \forall i$. This is in contradiction to Theorem 4.5 if we take our infinite sequence to be $\{q^i\}_{i=1}^\infty$. \square

Theorem 5.5. *For every $q \in \mathbb{Z}_+$, $\exists \gamma, \gamma' \in \mathcal{C}(S)$ so that $\gamma \equiv_q \gamma'$ but $\gamma \not\equiv_h \gamma'$.*

Proof: Let γ and γ' be from the construction in Theorem 5.1. If $\gamma \equiv_h \gamma'$ then γ and γ' can be oriented so that they represent the same homology class [12]. In our construction γ and γ' have different homology representatives, thus those curves can not be h -equivalent. \square

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